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# Linear programming applied to water distribution networks

## Programación lineal aplicada a redes de distribución de agua

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**Abstract**
In the present work, an algorithm for the optimization of water distribution networks is developed using linear programming in Visual Basic. In this type of application, the desired objective is to minimize the costs of the network, including its installation. These costs depend largely on the diameter that needs to be optimized. The proposed algorithm, when tested, yields several solutions; where the optimal value depends on the objective of the function of a particular application.
**key words:** linear programming, networks, pressure, water distribution

**Resumen**
En el presente trabajo, se desarrolla un algoritmo para la optimización de redes de distribución de agua usando programación lineal en Visual Basic. En este tipo de aplicación, el objetivo deseado consiste en minimizar los costos de la red, incluida su instalación. Estos costos dependen en gran medida del diámetro que se requiera optimizar. El algoritmo propuesto, al ser probado, arroja varias soluciones, en donde el valor óptimo, depende del objetivo de la función de una aplicación en particular.
**Palabras clave:** programación lineal, redes, presión, distribución de agua

## 1. Introduction

The first linear programming (LP) model was built in the 18th century to estimate astronomical data. LP models were again formulated during the Second World War to solve resource allocation and transport problems, and their use became popular during the second half of the 20th century, once an efficient algorithm and computers capable of perform the calculations quickly enough (Goberna, Jornet, y Puente, 2004).

In the practical problems that the current world faces, a multitude of factors (raw material, labor, transportation, available resources, economic levels, time, etc.) are involved, subject to multiple restrictions, with which you want to obtain maximum benefits or minimum costs. The part of mathematics with which this type of problem is solved is called linear programming, where one seeks to maximize or minimize an objective function.

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In the case of networks that transport water, the problem is to determine a set of diameters, such that a cost function is minimized (based on the length, diameter and cost of the pipes) subject to hydraulic, commercial restrictions, etc. (Araque, y Saldarriaga, 2005).

Next, linear programming is described, and especially, its application to problems related to water distribution networks, to finally develop an algorithm in Visual Basic and test it on a network, where its diameters are optimized to minimize costs.

## 2. Linear programming

Linear programming problems refer to the efficient use or distribution of limited resources to achieve the desired objectives. These problems are characterized by the large number of solutions that satisfy the basic conditions of each problem. The selection of a particular solution, as the best solution to a problem, will depend to some degree on the overall objective implicit in the statement of the problem. A solution that satisfies both the conditions of the problem and the given objective is called the optimal solution (Gass, 1996).

On the other hand, systems where there are more variables than equations have been called indeterminate. In general, indeterminate systems of linear equations do not have a solution, or they have an infinite number of solutions. An important method to obtain solutions to indeterminate systems of equations is to reduce the system to a set that contains as many variables as equations, that is, to a certain set. This can be accomplished by making an appropriate number of variables equal to zero (Gass, 1996).

It is important to note that in linear programming after verifying the possible decisions that can be made; this leads to identify the variables of the specific problem, it leads to determine a set of restrictions that are determined taking into account the nature of the problem in question. To finally calculate the cost / benefit, associated with each admissible decision; this involves determining an objective function that assigns to each possible set of values for the variables that determine a decision, a cost / benefit value. The set of all these elements defines the optimization problem (Castillo, Conejo, Pedregal, García, y Alguacil, 2002).

**Table 1**
Overview of some network optimization applications

| APPLICATIONS | PHYSICAL ANALOGY OF NODES | PHYSICAL ANALOGY OF ARCOS | FLOW |
|---|---|---|---|
| Communication system | Telephony, computer transmissions, satellites | Cables, fiber optics, microwave radio links | Transmission of voice messages, data, videos |
| Hydraulic systems | Pumps, lakes, reservoirs | Pipelines | Water, gas, fluids, fuels |
| Integrated computing circuits | Doors, registers, processors | Wired | Electric current |
| Mechanical systems | Connections | Lever, connecting rods, springs | Heat, energy |
| Transportation systems | Crossings, airports, stations | Roads, air routes, railways | Passengers, loads, vehicles, operators |

Source: w3.cnice.mec.es/eos/ Educational Materials / mem2003 / programming / - 8k –

In general, the fundamental problem of linear programming is the optimization of a linear expression or function, given certain limiting or restrictive conditions, also linear, that are expressed in the form of equations or inequalities (Escudero, 1976).

### 2.1. Structure of a mathematical model of linear programming

The elements that make up a linear programming model (Ponce, Solis, y Ulfe, 2011) are presented in detail below.

#### 2.1.1. Variables

They represent the information for decision making. Its values or results are determined by the model. They must be defined by expressing an action to be carried out and in a unit of measurement.

a) The variables according to their use can be:

- Decision variables: They are those that indicate the action or alternative that must be followed to achieve a certain objective.
- Auxiliary variables: They serve as support for calculating the decision variables. Such as inventory, deviation, clearance variables, etc.

b) The variables according to their domain, can be:

- Fractional (continuous): of more general use, when they are admissible in the practice of decision-making on actions or alternatives with results in fractional values.
- Integers: To express decisions on whole quantities, which do not admit fractional values, as they are impossible to put into practice.
- Binary: To express mutually exclusive or dichotomous decisions. Values 0 and 1 are used.
- Unrestricted sign: To express activity levels of two opposite directions.
- Negatives: they only assume values less than zero.

c) To represent the variables, identifiers are used, which, depending on the complexity and size of the model, can be done in two ways:

- Identifiers of a single index: when you want to express a single dimension of information of the alternatives. Example: suppose you have n alternatives to do only one type of work. So let Xi: amount of work to be done using alternative i. For i: 1, .. n.
- Multiple index identifiers: when you want to express more attributes or more than one dimension of information per alternative. Example: suppose that there are n alternatives to perform m types of work in or periods of time. Then, let Xi, j, k: quantity to be done using alternative i, for work j in period k. For i: 1, .. n j: 1,… m and k: 1, .. o.

#### 2.1.2. Parameters

It is all the information known a priori and invariable for the planning horizon of the model. They must be expressed in some metric unit of measurement. These parameters can be:

- Objective function coefficients: these are corresponding factors for each decision variable; such as profits, sales prices, costs, weights, priorities, etc .; expressed in a unitary way by product or activity.
- Technological coefficients: these are corresponding factors for each decision and restriction variable; such as consumption or demand ratio of product or activity on a resource.
- Resource coefficient: these are corresponding factors for each restriction; such as availability of a limited resource, which can be material, financial or human resources. They can also express a condition or quota to be met.

#### 2.1.3. Objective Function

It is an expression of variables, or sometimes, a variable that represents the objective to be achieved and expresses the quantification of the effectiveness of planning in a specific unit of measurement or metric. It can be in two ways:

- Maximize (Max): in a problem it could be required, to increase as much as possible the profits or income from sales, effectiveness of policies or alternatives, volume to be transported, probability of success, etc.
- Minimize (Min): in a problem you would like to reduce as much as possible: costs, losses, deviations from a condition or restriction, the completion time of a job, risk of a stock market operation, etc.

In the event that it is desired to cover several objectives simultaneously and often opposed, an adaptation of the formulation is used through programming by goals.

#### 2.1.4. Restrictions

They are relationships between variables and parameters that are represented by inequalities or equations, they result due to resource limitations or certain technical provisions of a given problem. Constraints are expressed in a unit homogeneous to both members of the inequality or equation. Because of them, the units of the variables, technological coefficients, resource coefficients that intervene in a restriction, must be expressed by means of conversions, generating a homogeneous unit.

## 3. Optimization of water networks

One of the first and most profitable applications of linear programming has been the formulation and resolution of the transport problem as a linear programming problem (Escudero, 1976). An example of transport is the optimization of networks that transport a fluid, in this case, water.

For this type of problem the following theorems must be considered (Escudero, 1976):

- Theorem 1: The problem has a possible solution.
- Theorem 2: Variables X are positive, otherwise it would not make sense.
- Theorem 3: Any possible basic solution has integer values.
- Theorem 4: There is always a possible solution, finite minimum.

When network optimization is carried out, it is desired to minimize network costs, varying four parameters that characterize the pipeline (diameter, material, length and the coefficient of minor losses) (Fernandez, y Saldarriaga, 2005).

For the purposes of this work, the diameter of the pipe was chosen as the parameter to be modified, so that it is possible to carry the water from the sources to each node, maintaining a pressure above the minimum allowed pressure (Fernandez, y Saldarriaga, 2005).

It is important to keep in mind that the problem of designing drinking water distribution networks is quite complicated, due to the non-linear relationship between flow and head loss (head) and the presence of discrete variables, such as the diameters of commercial pipes. Additionally, the cost function of the pipes also has a non-linear relationship with the diameters. In fact, this relationship has been shown to be intractable for which no deterministic method is known to solve it in a polynomial time (Araque, y Saldarriaga, 2005).

## 3.1. Equations

### 3.1.1. Objective Function - Total Cost

This function, as previously mentioned, represents the objective to be achieved, which in this case is to minimize costs (Samani & Mottaghi, 2006).

$$F=(D_N H_K)=f(D_N)+g(H_K)=\sum_{J=1}^{NPA} L_N CP_N(D_N)+\sum_{K=1}^{NR} CR_K(H_K) \tag{1}$$

This function depends on: , which represents the cost of the pipe and its installation depending on the diameters of the pipe DN; , which denotes the pressure costs of the generating means (tanks and pumps); , the length of the number of tubes N in the network; , unit cost of pipe N as a function of pipe diameter DN; , pressure cost of generators K which is a function of (level of the tank or total dynamic head of the pump) including investment, maintenance and operation; , number of tubes in the network; , number of pressure generators.

If we multiply the previous equation, by variables of unit zero and, the expression is as follows:

$$F=(D_N H_K)=\sum_{J=1}^{NPA} \; +\sum_{N=1}^{NP} L_{NJ} CP_{NJ}(D_N) X_{NJ}+\sum_{J=1}^{NPA}\sum_{K=1}^{NR} CR_{KM}(H_K) Y_{KM} \tag{2}$$

Where: NPA represents the number of commercially available tube sizes and NRA the number of tanks or pumps.

### 3.1.2. Restrictions

The following restrictions must be considered for each branch in the network (Samani & Mottaghi, 2006).

$$\sum_{i=1}^{NPAn} X_{NJ}=1 \qquad \sum_{M=1}^{NRA} Y_{KM}=1 \tag{3}$$

### 3.1.2.1 Pressure Restrictions

When optimizing, consideration should be given to the fact that when diameters are varied and nodal heads are calculated, the pressure at that point should be neither less than the minimum allowed, nor greater than the stipulated limit. The restrictions to the development of the optimization help to fulfill the previously described, are described below (Samani & Mottaghi, 2006):

$$\frac{p_i}{\gamma}=H_R-\Delta Z_{R-i}-\sum_{t=NHR}^{L_{R-}} hft \tag{4}$$

Where: $P_i/\gamma$ = pressure head at node i; AZR-i. = difference in elevation between the reference node and node i; HR = head at the reference node; $\sum_{t=NHR}^{L_{R-}} hft$ = sum of minor losses from the reference node and node i; NHR = number of tubes connected to the reference node in part R-i; LR-i = Length of all tubes included between R-i nodes.

Another pressure constraint to consider is limits, that is, the minimum and maximum pressure. This is the following:

$$\frac{p_{\min imo}}{\gamma}\le\sum_{M=1}^{NRA} H_{RM} Y_{RM}-\Delta Z_{R-1}-\sum_{j-1}^{NPA} \; \sum_{t=NHR}^{L_{R-1}} h_{ft} X_{tj}\le\frac{p_{\max imo}}{\gamma} \tag{5}$$

#### 3.1.2.2. Speed Restrictions

When optimizing the network, varying the diameters of the tubes and calculating the speed of the flow carried between nodes, as with the nodal pressure, speed limits must be met, that is, minimum and maximum flow speed. The restriction that helps to ensure that the flow is transported with an acceptable speed is the following (Samani & Mottaghi, 2006):

$$V\min \leq V_N \leq V\max \tag{6}$$

Vmin y Vmax, they correspond to the minimum and maximum velocity of the flow in the pipes of the network.

Now, multiplying by the unit variable zero, we have the following:

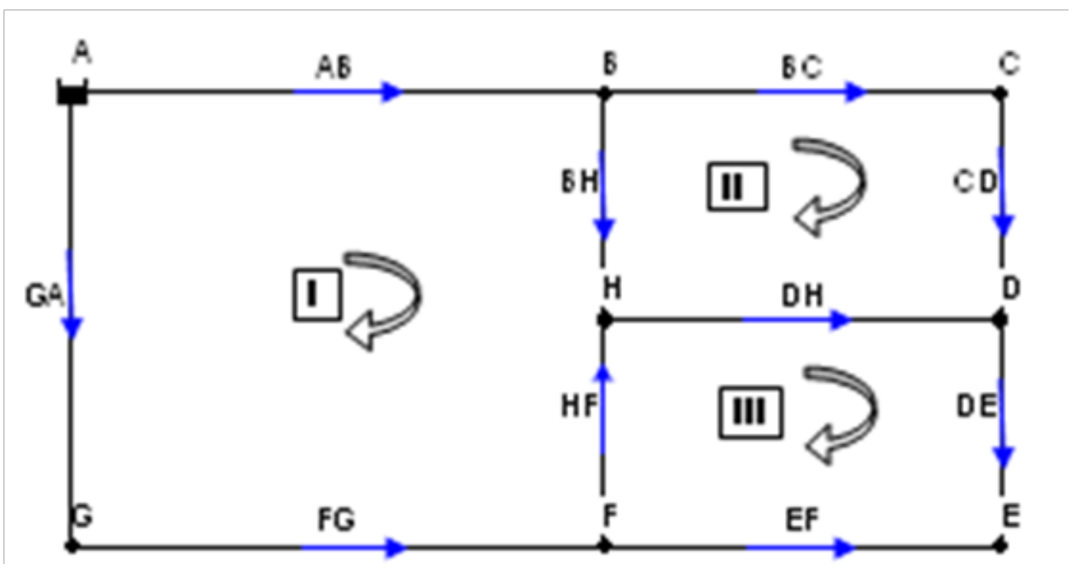
$$V\min \leq \sum_{j-1}^{NPA} \frac{Q_N}{\frac{\pi}{4} D_{NJ}{}^2} X_{NJ} \leq V\max \tag{7}$$

## 4. Results and discussion

For the optimization of water distribution networks, knowledge of the water supply through the network is required; the discharge can be determined initially only by applying the continuity equation, however, in closed networks, the flow discharge depends on the size of the pipe and the generating energy, which are unknown. To solve it, an iterative process is carried out through an algorithm.

In order to carry out the optimization analysis, the following closed network is solved step by step.

**Figure 1**
Closed network




For the network represented in Figure 1, the data is as follows:

**Table 2**
Characteristics of the network

| Tube | Length (m) | Km (m) | ks (m) |
|---|---|---|---|
| AB | 400 | 0,40 | 0,00006 |
| BC | 300 | 0,30 | 0,00006 |
| BH | 150 | 0,15 | 0,00006 |
| CD | 150 | 0,15 | 0,00006 |
| DH | 300 | 0,30 | 0,00006 |
| DE | 150 | 0,15 | 0,00006 |
| EF | 300 | 0,30 | 0,00006 |
| FG | 400 | 0,40 | 0,00006 |
| HF | 150 | 0,15 | 0,00006 |
| GA | 300 | 0,30 | 0,00006 |

The nodal demands are as follows:

**Table 3**
Demand on knots

| Node | Demand (L/s) |
|---|---|
| A | -230 |
| B | 50 |
| C | 20 |
| D | 20 |
| E | 40 |
| F | 30 |
| G | 40 |
| H | 30 |

The diameters used and the costs are:

**Table 4**
Unit costs per meter

| Diameter (mm) | Unit cost /m |
|---|---|
| 150 | 40.000 |
| 200 | 50.000 |

The allowable pressures are:

**Table 5**
Permissible pressures

| | **Pressure head** |
|---|---|
| Minimum | 60 m |
| Maximum | 100 m |

The height of the deposit and its costs are:

**Table 6**
Deposit costs

| Tank height (m) | Unit cost |
|---|---|
| 100 | 20.000.000 |
| 105 | 30.000.000 |

For the described network, the optimization analysis developed step by step is as follows:

1. The diameters of the pipe and the head of the water tank (level of the tank or pump head) will be assumed. For the scheme shown in Figure 1, the tank is located at node A.

2. The hydraulic analysis of the network is carried out, to obtain the head of the nodes, the discharge and the energy losses. This analysis is developed by the LINEAR THEORY method.

3. Then, the objective function is described:

$$F = (D_N H_K) = \sum_{J=1}^{NPA} + \sum_{N=1}^{NP} L_{NJ} CP_{NJ}(D_N) X_{NJ} + \sum_{J=1}^{NPA}\sum_{K=1}^{NR} CR_{KM}(H_K) Y_{KM} \quad (8)$$

For the analyzed case, this is determined as follows:

$$\begin{aligned}F = {} & L_{11}C_{11}X_{11} + L_{12}C_{12}X_{12} + L_{21}C_{21}X_{21} + L_{22}C_{22}X_{22} + L_{31}C_{31}X_{31} + \\ & L_{32}C_{32}X_{32} + L_{41}C_{41}X_{31} + L_{42}C_{42}X_{42} + L_{51}C_{51}X_{51} + L_{52}C_{52}X_{52} + \\ & L_{61}C_{61}X_{61} + L_{62}C_{62}X_{62} + L_{71}C_{71}X_{71} + L_{72}C_{72}X_{72} + L_{81}C_{81}X_{81} + \\ & L_{82}C_{82}X_{82} + L_{91}C_{91}X_{91} + L_{92}C_{92}X_{92} + L_{101}C_{101}X_{101} + \\ & L_{102}C_{102}X_{102} + C_{11}Y_{11} + C_{12}Y_{12}\end{aligned} \quad (9)$$

Replacing the network data, you have the following:

$$\begin{aligned}F = {} & 400*40.000X_{11} + 400*50.000X_{12} + 300*40.000X_{21} + \\ & 300*50.000X_{22} + 150*40.000X_{31} + 150*50.000X_{32} + \\ & 150*40.000X_{41} + 150*50.000X_{42} + 300*40.000X_{51} + \\ & 300*50.000X_{52} + 150*40.000X_{61} + 150*50.000X_{62} + \\ & 300*40.000X_{71} + 300*50.000X_{72} + 400*40.000X_{81} + \\ & 400*50.000X_{82} + 150*40.000X_{91} + 150*50.000X_{92} + \\ & 300*40.000X_{101} + 300*50.000X_{102} + 20.000.000Y_{11} + \\ & 30.000.000Y_{12}\end{aligned} \quad (10)$$

4. The network restrictions are reviewed. Since the Darcy-Weisbach equation is used, there are only pressure constraints on the nodes. These are:

<u>PRESSURE RESTRICTIONS</u>

$$\frac{p_{\min imo}}{\gamma} \leq \sum_{M=1}^{NRA} H_{RM} Y_{RM} - \Delta Z_{R-1} - \sum_{j-1}^{NPA} \sum_{t=NHR}^{L_{R-1}} h_{ft} X_{tj} \leq \frac{p_{\max imo}}{\gamma} \quad (11)$$

Replacing the corresponding values, for each node, we have:

Node B

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{11} - 12.096924Q^2X_{12} \leq 100 \tag{12}$$

Node C

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{11} - 12.096924Q^2X_{12} - 67.894377Q^2X_{21} - 12.096924Q^2X_{22} \leq 100 \tag{13}$$

Node D

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{11} - 12.096924Q^2X_{12} - 67.894377Q^2X_{21} - 12.096924Q^2X_{22} - 67.894377Q^2X_{31} - 12.096924Q^2X_{32} \leq 100 \tag{14}$$

Node E

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{11} - 12.096924Q^2X_{12} - 67.894377Q^2X_{21} - 12.096924Q^2X_{22} - 67.894377Q^2X_{31} - 2.096924Q^2X_{32} - 67.894377Q^2X_{41} - 12.096924Q^2X_{42} \leq 100 \tag{15}$$

Node F

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{101} - 12.096924Q^2X_{102} - 67.894377Q^2X_{91} - 12.096924Q^2X_{92} \leq 100 \tag{16}$$

Node G

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{101} - 12.096924Q^2X_{102} \leq 100 \tag{17}$$

Node H

$$60 \leq 100Y_{11} + 105Y_{12} - 67.894377Q^2X_{101} - 12.096924Q^2X_{102} - 67.894377Q^2X_{91} - 12.096924Q^2X_{92} - 67.894377Q^2X_{81} - 2.096924Q^2X_{82} \leq 100 \tag{18}$$

RESTRICTIONS OF ZERO VARIABLES

$$\begin{aligned}
X_{11} + X_{12} &= 1 \\
X_{21} + X_{22} &= 1 \\
X_{31} + X_{32} &= 1 \\
X_{41} + X_{42} &= 1 \\
X_{51} + X_{52} &= 1 \\
X_{61} + X_{62} &= 1 \\
X_{71} + X_{72} &= 1 \\
X_{81} + X_{82} &= 1 \\
X_{91} + X_{92} &= 1 \\
X_{101} + X_{102} &= 1
\end{aligned} \tag{19}$$

5. Using linear programming with the objective function described in subsection No.3, subject to restrictions, the size of the pipe is determined.

6. The results obtained are compared, in terms of pressure and network costs in the different iterations, and the best optimization is selected, that is, the result that best minimizes costs, guaranteeing hydraulic stability.

After several iterations, the best combination is as follows:

**Table 7**
Resume Table

| COMBINATION | Tube | Diameter (mm) | Pressure (m) | Total Cost |
|---|---|---|---|---|
| | AB | 0,2 | 100 | 131.000.000 |
| | BC | 0,15 | 70,38351697 | |
| | BH | 0,15 | 62,50699511 | |
| | CD | 0,15 | 64,09760164 | |
| Tank height | DH | 0,15 | 61,620397 | |
| 100 | DE | 0,15 | 64,09760164 | |
| | EF | 0,15 | 60,74101485 | |
| | FG | 0,15 | 64,11439135 | |
| | HF | 0,15 | 64,07362549 | |
| | GA | 0,2 | 88,1830092 | |

## 5. Conclusions

Linear programming is the study of mathematical models concerning the efficient allocation of limited resources in known activities, with the objective of satisfying the desired goals (such as maximizing benefits or minimizing costs). In linear programming MAXIMIZE or MINIMIZE linear functions. Solutions to indeterminate systems of equations are aimed at reducing the system to a set containing as many variables as equations, that is, to a given set, making an appropriate number of variables equal to zero. Excellent optimization depends on the number of iterations that are performed. When calculating pressure heads, the loss heads in the tubes must be considered. When performing the optimization analysis, it should be clear that the structure of the model requires an objective function declaration, variables and constraints. When performing network optimization analysis, it must be verified that it complies both hydraulically and economically.

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